PHYSICAL REVIEW E **70**, 035105(R) (2004)

Clogging and self-organized criticality in complex networks

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(Received 19 December 2003; published 22 September 2004)

We propose a simple model that aims at describing, in a stylized manner, how local breakdowns due to imbalances or congestion propagate in real dynamical networks. The model converges to a self-organized critical stationary state in which the network shapes itself as a consequence of avalanches of rewiring processes. Depending on the model's specification, we obtain either single-scale or scale-free networks. We characterize in detail the relation between the statistical properties of the network and the nature of the critical state, by computing the critical exponents. The model also displays a nontrivial, sudden collapse to a complete network.

DOI: 10.1103/PhysRevE.70.035105 PACS number(s): 89.75.Hc, 05.65.+b, 45.70.Ht

Complex networks underlying many social and technological systems is a subject of booming recent interest [1,2]. On one hand, the structure of such networks has nontrivial properties [1], which dramatically influences the nature of processes taking place on them (see, e.g., [3,4]). On the other hand, the network's structure constrains in a peculiar way the growth [5] and evolution [6–10] of the network itself. This calls for an extension of statistical physics, which traditionally studies collections of dynamical variables interacting through a fixed network, to systems where the network of interactions itself becomes a dynamical variable.

Here we focus on dynamical networks where links do not represent physical bonds, but rather relationships or communication channels. Reference [3] suggests that a structured network of communications aimed at solving problems or carrying out specific functions is a crucial feature of firms and organizations in general. A router's table in the Internet is also an example of a node in a dynamic communication network [11]. The network of financial institutions, linked by mutual contracts and loans, provides a further example of a dynamic network [12]. Beyond "static" design problems, such as, e.g., minimizing congestion [3] or redistributing optimally the loads [13]; systems of this type also pose "dynamical" problems such as how and to what extent do congestion or breakdown events propagate through the system.

Here we address these problems in a dynamic network subject to two competing forces: on one side, there is a drive toward increasing complexity, by, e.g., adding new links, because the system performs more efficiently its functions as it becomes more densely interconnected. On the other, the resulting increase in complexity may bring about conflicting constraints, imbalances, or congestion problems, which may cause a local breakdown of the network. A local breakdown may engender a readaptation in its neighborhood, which may inadvertently cause the breakdown to propagate further on the network.

For example, a change in some router's table in the Internet, which is meant to improve efficiency, may inadvertently cause congestion at some node downstream. This may trigger several other changes in that local neighborhood, as routers try to avoid the congested nodes. But these changes may, in their turn, cause further congestion elsewhere, and the problem may expand even further, as an avalanche, to a wider region. The stipulation of a contract between two institutions, which is, in principle, beneficial to both, may also increase their operative constraints, making them less adaptable to a changing environment and hence more exposed to the risk of bankruptcy. The failure of one institution likely induces a rearrangement of the institutions linked with it and perhaps engenders effects which propagate further across the network [12]. Similar phenomena may take place in social or trade networks.

Rather than trying to model, in a realistic manner, one of the problems just discussed, we focus on a simple model of network dynamics that captures the two main ingredients discussed above: a slow dynamics where links are added to the network and a fast relaxation dynamics of avalanche events. The motivation for this choice is that the detailed understanding of the behavior of a simple model with these features may be the basis or at least a guide for addressing more complex and realistic situations, such as those discussed above. Our main finding is that such systems can self-organize close to a critical point where each modification of the network's architecture can have unforeseeable consequences which possibly affect a wide region of the system [14]. This may have some bearing on the intermittence of internet traffic [20] or on the nature of financial crises and recessions. While several other models have been proposed that exhibit phase transition [6,10] or critical behavior [7–9], none exhibit the peculiar features of self-organized critical systems [14].

We start from a empty network of N nodes in which every node i is assigned a fitness f_i drawn from a probability distribution $\rho(f)$. Let F_i be the set of neighbors of i, and $k_i = |F_i|$ be the number of neighbors of i. At every time step a link is added between two previously unconnected random nodes i and $j \notin F_i$. With probability f_j nothing happens, whereas with probability $1-f_j$ the node j becomes unstable or congested and it "topples." As a result all its links (including that with i) are rewired to randomly chosen nodes, i.e., for any $h \in F_j$, a node $l \notin F_h$ is chosen at random and the link jh is rewired to hl. In its turn, with probability $1-f_l$, also node l may become unstable and topple. Hence, toppling of node j may start an avalanche of toppling events which propagates through the network rearranging it. More precisely, unstable nodes are selected for toppling sequentially in a random order, until no unstable node remains. Unstable nodes, after they topple, remain unconnected from the network [15] and are assigned a new fitness value drawn from the distribution $\rho(f)$. Hence, toppling is equivalent to replacing the unstable node with a new one.

In order to stabilize the network and reach a stationary state, we introduce the dissipation of links: at each toppling event, with probability λ, all the links of the unstable node are removed from the network. Note that without dissipation the number of links would increase in time until the complete graph is reached. The complete graph $(j \in F_i \forall i,j,i \neq j)$ is an absorbing state of the dynamics, because no link can be added to it.

The distribution $\rho(f)$ is the only parameter of the model. An alternative class of models can be defined by specifying the probability u_k that a node with k neighbors becomes unstable upon addition of a further link. A relation between the two models is possible along the lines of Ref. [16], in the limit $\lambda \to 0$ (no dissipation). Then the probability to find a node with k neighbors and $f_i \in [f, f+df)$ is $\rho(f|k)df \propto f^k \rho(f) df$, where f^k is the probability that a node with $f_i = f$ has k neighbors. Then a model with

$$u_k = \int_0^1 df(1-f)\rho(f|k) = \frac{\int_0^1 df(1-f)f^k\rho(f)}{\int_0^1 df \quad f^k\rho(f)} \qquad (1)$$

is completely equivalent to one specified in terms of $\rho(f)$, in the limit $\lambda \to 0$. The dependence on k of u_k reflects the fact that k_i and f_i are positively correlated because nodes with higher fitness have a smaller chance of becoming unstable. For convenience we shall refer mostly to models specified in terms of u_k, using Eq. (1) to translate the results in the original model.

Our results can be summarized as follows: (i) when u_k decays faster than $1/k$ there is a critical λ_c such that the network evolves toward a complete graph for $\lambda < \lambda_c$. The same happens for $u_k \simeq b/k (k \gg 1)$ and $b < 3/2$, whereas when $b > 3/2$ or when u_k decays slower than $1/k$, the collapse takes place only in finite networks. Indeed we find complete graphs only for $\lambda < \lambda_c \sim N^{-\gamma}$, where $\gamma = (2b-3)/(b-1)$ for $3/2 < b < 2$ and $\gamma = 1$ otherwise. (ii) the noncollapsed phase $\lambda > \lambda_c$ is characterized by an uncorrelated random network [17] with finite average degree and a degree distribution p_k that depends on $\rho(f)$ (or u_k) (see Fig. 1). In particular, if u_k decays slower than $1/k$ then p_k decays faster than any power, whereas if $u_k \simeq b/k$ then $p_k \sim k^{-b}$. (iii) The dynamics converges to a stationary sequence of avalanches of rewiring processes with a power law distribution $P(s) \sim s^{-\tau}$ of sizes (see Fig. 3). As in Ref. [16], we shall define the size s of an avalanche as the number of toppling events that it causes. The exponent τ takes the mean-field value [18] $\tau = 3/2$ when u_k decays slower than $2/k$, whereas $\tau = 1 + 1/b$ if $u_k \simeq b/k$ with $3/2 < b < 2$.

The collapse to a complete graph, where congestion is minimal, is reasonable, given that the network is trying to adapt by avoiding congestion. The nontrivial issue on which we shall concentrate mostly, is related to the self-organized critical state. In view of the special role played by the case $u_k \sim 1/k$, we focus on the following simple forms of $\rho(f)$ and to the corresponding u_k:

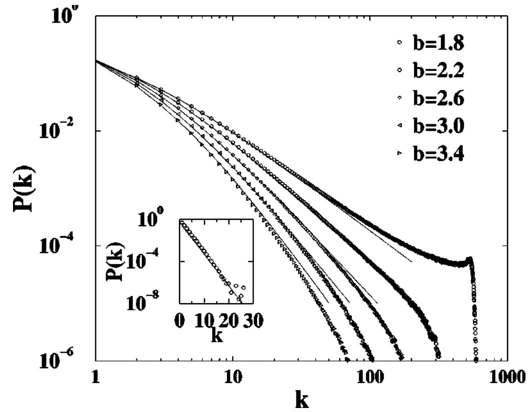

FIG. 1. Degree distribution p_k for the network of $N = 10^4$ nodes (averaged over $300N$ time steps after equilibration) for the model of Eq. (2) and $b = 1.8, \ldots, 3.4$ and $\lambda = 5 \times 10^{-3}$ and for $u_k = u - 0.5$ (inset). The solid lines correspond to the theoretically expected p_k in the limit $\lambda \to 0$ [Eq. (8) and $p_k = u(1-u)^k$, respectively]. The peak for $b = 1.8$ and $k \gg 1$ is a precursor of the phase transition to the complete graph discussed in the text.

$$\rho(f) = b(1-f)^{b-1}, \quad u_k = \frac{b}{b+k+1}. \qquad (2)$$

In order to shed light on the model's behavior, let us derive an equation for the avalanche distribution. Define s_k as the avalanche size originating from an unstable node with k neighbors. This is a random variable that can be decomposed as follows:

$$s_k = 1 + d \sum_{j=1}^{k+1} v_{k_j} s_{k_j} \qquad (3)$$

into the contribution of the unstable node and those of the avalanches s_{k_j} ensuing from its neighbors, with k_j being the number of neighbors of the jth neighbor. Note that the sum runs over $k+1$ links as it includes the link which caused the instability and the k preexisting neighbors. In Eq. (3) $d = 0, 1$ describes the effect of dissipation with $P(d=0) = \lambda = 1 - P(d=1)$, whereas v_{k_j} takes value $v_{k_j} = 0$ if the rewiring of the link to the jth neighbor causes no further toppling, and $v_{k_j} = 1$ otherwise. Hence, $P(v_l = 1) = 1 - P(v_l = 0) = u_l$.

Now we can write the generating function $\phi_k(z) = \langle z^{s_k} \rangle$ of the probability $P(s|k)$ to have an avalanche of size s given that the initiator node has k neighbors. From this, it is easy to find the generating function of the distribution $P(s)$ of avalanche sizes $\chi(z) \equiv \sum_s P(s) z^s = 1/\overline{u} \sum_{k=0}^{\infty} p_k u_k \phi_k(z)$ with $\overline{u} = \sum_k p_k u_k$. After some algebra, using the fact that the rewired nodes are chosen randomly in the network, Eq. (3) leads us to

$$\chi(z) = z\left[\lambda + \frac{1-\lambda}{\overline{u}}\sum_{k=0}^{\infty} p_k u_k[1-\overline{u}+\overline{u}\chi(z)]^{k+1}\right], \quad (4)$$

which is a nonlinear self-consistent equation for $\chi(z)$. Note that $\chi(1)=1$ as it should and that

$$\chi'(1) = \langle s\rangle = \frac{1}{1-(1-\lambda)\langle(k+1)u\rangle}. \quad (5)$$

Apart from the parameters λ and u_k, Eq. (2) depends also on the stationary-state degree distribution p_k. Hence, before discussing Eq. (3) further, we need to elaborate on the nature of the stationary state. A necessary condition in order to be at the stationary state is that the total number of links $K(t)$ present in the network is constant on average. $K(t)$ increases by one for the random addition of a new link and is reduced by the total number of links Λ dissipated during the avalanche that follows. Here

$$\Lambda = v\sum_{i=1}^{s} d_i(k_i+1), \quad (6)$$

where $v=1$ if the chosen site is unstable and $v=0$ otherwise, and $d_i=1$ if dissipation occurs at the toppling site i, otherwise $d_i=0$. Consequently, v and d_i have average values $\overline{u}$ and λ and thus $\langle\Lambda\rangle=\lambda\langle(k+1)u_k\rangle\langle s\rangle$. Then, stationarity $\langle\Lambda\rangle=1$ and Eq. (5) imply that $\langle(k+1)u_k\rangle=1$. As a by-product, we find $\langle s\rangle=1/\lambda$, in perfect agreement with numerical simulations.

The stationary degree distribution p_k can be found by quantifying the Markov chain of possible transitions $k_i\to k_i'$ during the dynamics. In the limit $\lambda\to 0$ and $N\to\infty$, where we can neglect dissipation and finite-size effects, there are only two processes that take place on each node: $k_i\to k_i+1$, with probability $1-u_{k_i}$ and $k_i\to 0$ with probability u_{k_i}. Then, in the stationary state, p_k satisfies

$$p_{k+1} = (1-u_k)p_k. \quad (7)$$

Taking the sum of Eq. (7) on k we find $p_0=\langle\overline{u_k}\rangle$, which means that the fraction of sites with no neighbor is equal to the probability that a node becomes unstable. Furthermore, multiplying Eq. (7) by $k+1$ and taking the sum over k, we recover the stationary condition $\langle(k+1)u_k\rangle=1$. In the simplest case $u_k=u$ for all k, we find $p_k=u(1-u)^k$, whereas with Eq. (2) we find

$$p_k = (b-1)\frac{\Gamma(b)\Gamma(k+1)}{\Gamma(b+k+1)} \sim k^{-b}, \quad (8)$$

where the asymptotic power-law behavior holds for $k\gg 1$. Notice that $\langle k\rangle=1/(b-2)$ diverges when $b\to 2^+$ and that there is a finite fraction $\overline{u}=1-1/b$ of unconnected nodes. Still, the network has a giant connected component for $b<7$ [17]. Equation (7) yields a p_k which decays faster than any power if u_k decays less slowly than $1/k$, or if it increases. Conversely, if u_k decays faster than $1/k$, we find that p_k is not normalizable for $N\to\infty$. Numerical simulations (see Fig. 1) fully support this picture, even though the effects of dissipation and finite size are clearly evident for $k\gg 1$.

The neglect of dissipation, when N is finite, is a reasonable approximation if nodes with maximal degree $k_i=N-1$ are not stable. A node connected to all neighbors, cannot receive further links and hence cannot become unstable. Its degree decreases only if dissipation occurs at a node connected to them. The rate of this process, for a node with degree k, is $\lambda k/\langle k\rangle$. Hence, if $\lambda N/\langle k\rangle\gg 1$, nodes with $k\sim N$ decay very fast and the only effect of dissipation is to introduce a cutoff $k_c\sim 1/\lambda$ in the distribution p_k. When $\lambda N/\langle k\rangle \sim 1$ we expect a transition (close) to the complete graph $k_i=N-1$ for all i. When $\langle k\rangle$ is finite, i.e., for $b>2$ or for u_k, which decays slower than $1/k$, the collapse to a complete graph takes place for $\lambda<\lambda_c\sim N^{-1}$. When $3/2<b\leqslant 2$ the average degree $\langle k\rangle\sim N^{(2-b)/(b-1)}$ diverges with the system size. Then the decay rate of totally connected nodes is $\sim\lambda N^{\gamma}$, with $\gamma=(2b-3)/(b-1)$ and the collapse to a complete graph takes place for $\lambda<\lambda_c\sim N^{-\gamma}$. In both cases (i.e., for $b>3/2$) for any $\lambda<1$, it is always possible to take N large enough to make the decay rate λN^{γ} large enough so that finite-size effects can be neglected. But when $b<3/2$ this is no longer true because $\langle k\rangle\sim N$ and the decay rate of completely connected nodes remains finite ($\gamma=0$) even when $N\to\infty$. Beyond a finite dissipation rate λ_c, the network collapses to the complete graph.

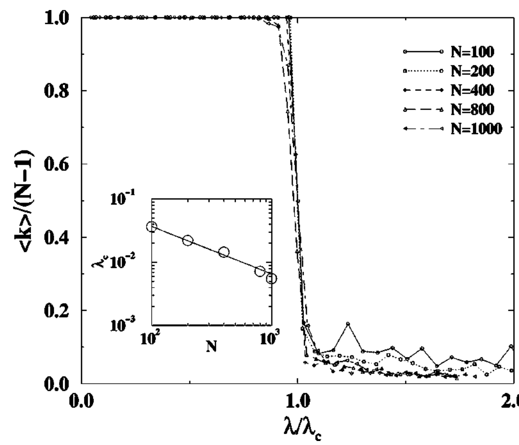

FIG. 2. Average degree as a function of λ for different system sizes and $b=1.8$. The value of λ_c at which the transition takes place is plotted in the inset against N. The full line is the theoretical prediction $\lambda_c\propto N^{-\gamma}$ with $\gamma=0.75$ for $b=1.8$.

Figure 2 fully confirms the theoretical insight discussed above. When $\lambda>\lambda_c$, where $\lambda_c\sim N^{-\gamma}$ (see inset), the dynamics reaches a network with finite average degree, whereas for $\lambda<\lambda_c$ a collapse to the complete graph is observed, with a transition that is sudden and discontinuous. In the case in which the $u_k=u$ we also observe a transition from a finite average connectivity to a average connectivity of order N. The transition occurs for values of $\lambda_c\sim N^{-1}$, but the transition is rather smooth.

Having discussed the stationary state, let us go back to the Eq. (3) for the avalanche-size distribution. We focus on the region $1\gg\lambda\gg N^{-\gamma}$, where the network is not densely connected and dissipation effects are weak. Anticipating that the avalanche-size distribution acquires a cutoff $s_c\sim\lambda^{-\sigma}$, for some exponent $\sigma>0$, we postulate the scaling form $P(s)\simeq s^{-\tau}\Phi(s\lambda^{\sigma})$ with finite $\Phi(0)$ and $\Phi(x)\to 0$ faster than any power as $x\to\infty$. Such a scaling hypothesis is fully corrobo-

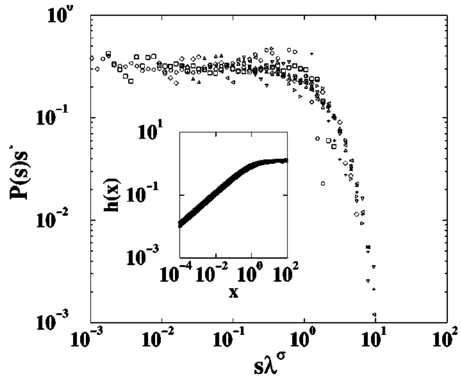

FIG. 3. Data collapse of the avalanche distribution $P(s)$ with $s>7$ for networks of $N=10^3$ nodes averaged over $200N$ time steps after equilibration. Here $b=2.5$ and $\lambda=0.01, 0.02, 0.04, 0.06, 0.18, 0.31$. The data collapse was done taking the theoretical values $\tau=3/2$ and $\sigma=2$ of the exponents. Inset: the scaling function $h(x)$ for the same data set.

rated by numerical results, as shown in Fig. 3. This corresponds [19] to an analogous scaling form

$$\chi(z) \simeq 1-(1-z)^{\tau-1}h\left(\frac{1-z}{\lambda^{\sigma}}\right) \quad (9)$$

for the generating function for $\lambda \ll 1$ and $1-z \sim \lambda^{\sigma}$. Setting for convenience $1-z=x\lambda^{\sigma}$, asymptotic analysis for $\lambda \ll 1$ shows that the leading orders of Eq. (4) are

$$\lambda^{1+\sigma(\tau-1)}x^{\tau-1}h(x) = \lambda^{\sigma}x - c\lambda^{\sigma\beta(\tau-1)}x^{\beta(\tau-1)}h^{\beta}(x), \quad (10)$$

where $\beta=2$, and $c=(b-1)/[b(b-2)]$ for $b>2$ while $\beta=b$, $c=[\pi b^{b-1}(1-b)^{b}]/\{\sin[\pi(2-b)]\}$ for $b<2$. Note that $c \sim 1/|b-2|$ diverges when $b \to 2$ is approached from both sides. Dividing Eq. (10) by λ^{σ} and taking the scaling limit $\lambda \to 0$ with x finite, we find a nontrivial result with all three terms finite if we choose

$$\tau = 3/2, \quad \sigma = 2 \quad \text{for } b > 2 \quad (11)$$

$$\tau = 1 + 1/b, \quad \sigma = b/(b-1) \quad \text{for } b < 2. \quad (12)$$

and the scaling function is the inverse of $x(h)=h^{\beta}/[1-ch^{\beta}]^{\beta}$. In particular, $h(x) \to c^{1/\beta}$ for $x \to \infty$ and $h(x) \sim x^{1/\beta}$ for $x \ll 1$. For $b>2$ the solution coincides with that of other mean-field models [19] $h(x)=[\sqrt{c/x+4}-\sqrt{c/x}]/(2c)$ and perfectly matches numerical simulations for a range of values of λ (see the inset of Fig. 3). It is easy to check that the model with u_k decaying slower than $1/k$ falls in the $\beta=2$, $\tau=3/2$, $\sigma=2$ universality class.

In conclusion, we have shown how a slow growth dynamics and a fast relaxation through avalanche events can generate a dynamical network with given degree distribution. The stationary state is critical in the sense that avalanches of all sizes occur, and it is reached spontaneously without fine tuning of external parameters as long as the dissipation rate is larger than a given threshold $\lambda_c \sim N^{-\gamma}$. For smaller dissipation rates, the network collapses to the complete graph. While the detailed solution depends on the particular simplicity of the model chosen, the generic picture may apply to a wider class of systems and capture some features of the nonlinear and intermittent behavior of real systems.

We acknowledge F. Vega-Redondo for useful comments and discussions. This work is supported in part by the European Community's Human Potential Programme under Contract No. HPRN-CT-2002-00319, STIPCO.

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